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Half-Space Problems for the Boltzmann Equation: A Survey

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This paper reviews recent mathematical results on the half-space problem for the Boltzmann equation. The case of a phase transition is discussed in detail.

KEY WORDS: Boltzmann equation, Boundary layers, Half-space problem.

1. PRESENTATION OF THE HALF-SPACE PROBLEM

Consider the steady state of a rarefied, monatomic gas in a half-space; by an appropriate choice of coordinates, we assume that this half-space is

$$\mathbf{R}^3_+ := \{(x, y, z) \in \mathbf{R}^3 \mid z \geq 0\} .$$

The state of the gas is described by its phase-space density (or velocity distribution function)

$$F \equiv F(x, y, z, v_x, v_y, v_z) \geq 0$$

that is the density of particle located at the position (x, y, z) with the velocity $v = (v_x, v_y, v_z) \in \mathbf{R}^3$. Everywhere in this paper, we assume that the steady state considered has slab-symmetry, meaning that the phase-space density is independent of the variables x, y; however, it does depend on the variables v_x, v_y. In other words,

$$F \equiv F(z, v), \quad z \geq 0, \quad v = (v_x, v_y, v_z) \in \mathbf{R}^3 . \tag{1.1}$$

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The phase-space density F satisfies the Boltzmann equation, which, in the case of slab-symmetry as above and in the absence of external forces (such as gravity), reduces to

$$v_z \partial_z F = \mathcal{B}(F, F), \quad z > 0, \quad v \in \mathbf{R}^3, \tag{1.2}$$

where $\mathcal{B}(F, F)$ is the Boltzmann collision integral. In the sequel, we shall assume that the molecular interaction is well described by the hard-sphere model, meaning that $\mathcal{B}(F, F)$ is of the form

$$\mathcal{B}(F, F)(v) = \iint_{\mathbf{R}^3 \times \mathbf{S}^2} (F' F'_* - F F_*) |(v - v_*) \cdot \omega| dv_* d\omega, \tag{1.3}$$

by choosing a unit of length so that $d_m^2/2m$ is unity, where d_m is the diameter of a molecule and m is the mass of a molecule. In the formula (1.3), the notations F_*, F' and F'_* designate as usual $F(v_*)$, $F(v')$ and $F(v'_*)$ respectively, where the velocities v' and v'_* are given in terms of v, v_* and ω by the formulas

$$\begin{aligned} v' &= v - (v - v_*) \cdot \omega\omega, \\ v'_* &= v_* + (v - v_*) \cdot \omega\omega. \end{aligned} \tag{1.4}$$

In order for the half-space Boltzmann equation (1.2) to define a unique solution F, some boundary conditions must be prescribed at least at $z = 0$ and also possibly at infinity (i.e. for $z \to +\infty$). As will be seen in our discussion below, the boundary condition at $z = 0$ may in some cases (or may not) influence the asymptotic behvior of F as $z \to +\infty$.

Another way of considering the same question is as follows. Assume the gas is in thermal equilibrium at infinity, meaning that its phase space density is a Maxwellian defined by its temperature $\theta_\infty > 0$, pressure $p_\infty > 0$ and the flow velocity of the gas $u_\infty \in \mathbf{R}^3$. In the sequel, we denote this Maxwellian state

$$\mathcal{M}_{(\rho_\infty, u_\infty, \theta_\infty)} = \frac{\rho_\infty}{(2\pi\theta_\infty)^{3/2}} \exp\left(-\frac{|v - u_\infty|^2}{2\theta_\infty} \right), \tag{1.5}$$

where the density ρ_∞ is related to the pressure by the ideal gas law

$$p_\infty = \rho_\infty \theta_\infty .$$

(Henceforth, the temperature is measured in units of energy, so that the specific gas constant, or the Boltzmann constant divided by the mass of a molecule, is 1). Thus, instead of discussing the influence of a given boundary condition at $z = 0$ on the state of the gas at infinity, one can equivalently seek all the boundary data at $z = 0$ for which the problem (1.2) has a unique solution F satisfying

$$F(z, v) \to \mathcal{M}_{(\rho_\infty, u_\infty, \theta_\infty)} \text{ as } z \to +\infty . \tag{1.6}$$

In our discussion of the half-space problem (1.2) we shall mainly consider the latter formulation. If we think of (1.2) as a dynamical system with z being the time variable, this formulation is equivalent to seeking the stable manifold of a given Maxwellian state at infinity. However, viewing (1.2) as a dynamical system is not entirely appropriate, as we shall see below.

The interest for half-space problems such as (1.2) mainly stems from their role in the asymptotic behavior of the solution of boundary-value problems of the Boltzmann equation for small Knudsen numbers. That is, the solution of half-space problems provides the boundary conditions for the fluid-dynamic-type equations and Knudsen-layer corrections to the solution of the fluid-dynamic-type equations in a neighborhood of the boundary (see Sone refs. 22 and 23). For that reason alone, this subject has received a lot of attention, and problems such as (1.2) are among the best documented of all steady problems involving the Boltzmann equation. However, as we shall see below, there remain several fascinating open questions in the theory of half-space problems such as (1.2). Carlo Cercignani greatly contributed to that theory,[3,9,11] and formulated some of the remaining open problems in this direction.[10] We offer him this modest contribution in recognition of his leading role in the development of kinetic theory in the past 40 years.

2. NUMERICAL AND ASYMPTOTIC ANALYSIS OF THE PHASE TRANSITION PROBLEM

2.1. Range of Existence of a Solution

The half-space problem (1.2) with (1.6) has only a trivial uniform solution that accommodates the boundary wall when the wall is a simple boundary, where the mass flux across the boundary vanishes, as given by Remark 2 in Sec.5. In the case where the wall is made of the condensed phase of the gas and evaporation or condensation takes place there, the range of the parameters where a steady (or time-independent) solution exists shows an interesting feature. The problem is studied analytically[20,26] and numerically (refs. 1, 2, 25, and 29, etc.). The numerical works on the basis of the BKW equation[3] and the complete condensation condition on the wall clarify the comprehensive feature of the existence range of the solution. The complete condensation condition at $z = 0$ is given by

$$F(z, v) = \mathcal{M}_{(\rho_w, 0, \theta_w)} = \frac{\rho_w}{(2\pi\theta_w)^{3/2}} \exp\left(-\frac{|v|^2}{2\theta_w}\right) \quad (v_z > 0), \tag{2.1}$$

where θ_w is the temperature of the wall (or the condensed phase) and ρ_w is the saturation gas density at temperature θ_w.

[3] The BKW equation is a model equation introduced in refs. 6 and 34, where the collision term $\mathcal{B}$ of (1.2) is simplified (see, e.g., ref. 22).

For the convenience of explanation, p_∞ and p_w ($= \rho_w\theta_w$) will be used instead of ρ_∞ and ρ_w, and the components of the flow velocity u_∞ at infinity are indicated by $(u_{x\infty}, u_{y\infty}, u_{z\infty})$. The component $u_{y\infty}$ can be put zero without loss of generality. The Mach numbers $M_{n\infty}$ and $M_{t\infty}$ with sign, corresponding to $u_{z\infty}$ and $u_{x\infty}$, are defined by

$$M_{n\infty} = u_{z\infty} \Big/ \left(\frac{5}{3}\theta_\infty\right)^{1/2}, \quad M_{t\infty} = u_{x\infty} \Big/ \left(\frac{5}{3}\theta_\infty\right)^{1/2}. \tag{2.2}$$

The basic equation and boundary condition being expressed in properly chosen non-dimensional variables, the boundary-value problem is found to be characterized by the four parameters $M_{n\infty}$, $M_{t\infty}$, p_∞/p_w, and θ_∞/θ_w.

The problem is first studied as a time-evolution problem of the Boltzmann equation with time-derivative term for a large number of initial and boundary data. From detailed investigation of the long-time behavior of the solutions, we found that the time-independent solution exists only in a limited range of parameters $M_{n\infty}$, $M_{t\infty}$, p_∞/p_w, and θ_∞/θ_w. Then, on the basis of a further larger number of solutions, the range of the parameters where a time-independent solution exists is determined. The result is summarized in the rest of this subsection.

When $M_{n\infty} \geq 0$, a solution exists when and only when the parameters satisfy the relations

$$p_\infty/p_w = h_1(M_{n\infty}), \quad \theta_\infty/\theta_w = h_2(M_{n\infty}), \quad M_{t\infty} = 0 \quad (0 \leq M_{n\infty} \leq 1), \tag{2.3}$$

where $h_1(M_{n\infty})$ and $h_2(M_{n\infty})$ are decreasing functions of $M_{n\infty}$ with $h_1(0) = h_2(0) = 1$. No solution exists for $M_{n\infty} > 1$. The solution is determined only by the parameter $M_{n\infty}$, and it is a subsonic or sonic evaporating flow or the uniform equilibrium state at rest with pressure p_w and temperature θ_w.

When $M_{n\infty} < 0$, the character of the solution is classified into two cases. If $-1 < M_{n\infty} < 0$ (a subsonic condensing flow), a solution exists when and only when the parameters satisfy the condition

$$p_\infty/p_w = F_s(M_{n\infty}, |M_{t\infty}|, \theta_\infty/\theta_w), \tag{2.4}$$

where F_s is a decreasing function of $M_{n\infty}$ with $F_s(0_-, |M_{t\infty}|, \theta_\infty/\theta_w) = 1$ (see Fig. 1).[4] The solution is determined by the three parameters $M_{n\infty}$, $|M_{t\infty}|$, and θ_∞/θ_w with trivial difference for positive or negative $M_{t\infty}$. If $M_{n\infty} \leq -1$ (a supersonic condensing flow), a solution exists when the parameters satisfy the

[4] According to direct simulation Monte-Carlo computation for a hard-sphere gas for $|M_{t\infty}| = 0$ and $\theta_\infty/\theta_w = 0.5$, 1, and 2 (Sone-Sasaki [unpublished]), the relative differences of its data p_∞/p_w from that of the BKW model are less than 1% except in the range $-0.9 \geq M_{n\infty} > -1$ at $\theta_\infty/\theta_w = 0.5$, where the differences are bounded by 5%.

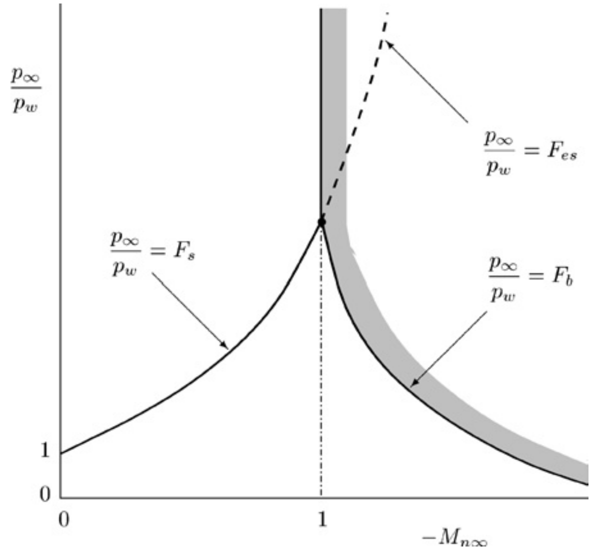

Fig. 1. A diagram of the range of existence of the solution of a condensing flow. The section $\theta_\infty/\theta_w =$ const. and $M_{t\infty} =$ const. of the parameter space $(M_{n\infty}, M_{t\infty}, p_\infty/p_w, \theta_\infty/\theta_w)$ where the solution exists is shown. A solution exists on the surface $p_\infty/p_w = F_s(M_{n\infty}, |M_{t\infty}|, \theta_\infty/\theta_w)$ and in the region $p_\infty/p_w \geq F_b(M_{n\infty}, |M_{t\infty}|, \theta_\infty/\theta_w)$, where the equal sign is applied only to the case $M_{n\infty} = -1$. A supersonic Knudsen-layer-type solution exists on the surface $p_\infty/p_w = F_{es}(M_{n\infty}, |M_{t\infty}|, \theta_\infty/\theta_w)$.

inequality

$$\begin{aligned} &p_\infty/p_w > F_b(M_{n\infty}, |M_{t\infty}|, \theta_\infty/\theta_w) \ (M_{n\infty} < -1), \\ &p_\infty/p_w \geq F_b(-1, |M_{t\infty}|, \theta_\infty/\theta_w), \end{aligned} \tag{2.5}$$

where F_b is an increasing function of $M_{n\infty}$ (see Fig. 1). That is, we can choose all the four parameters as long as they satisfy the above inequality. The two functions F_s and F_b approach the same limit as $M_{n\infty}$ approaches -1 from their regions:

$$\lim_{M_{n\infty}\to -1_+} F_s(M_{n\infty}, |M_{t\infty}|, \theta_\infty/\theta_w) = \lim_{M_{n\infty}\to -1_-} F_b(M_{n\infty}, |M_{t\infty}|, \theta_\infty/\theta_w). \tag{2.6}$$

Further, a point on $p_\infty/p_w = F_b(M_{n\infty}, |M_{t\infty}|, \theta_\infty/\theta_w)$ is related to some point on $p_\infty/p_w = F_s(M_{n\infty}, |M_{t\infty}|, \theta_\infty/\theta_w)$ by the Rankine–Hugoniot relation of a plane shock wave or the shock condition.[(13)]

The bounds of the functions h_1, h_2, F_s, and F_b are discussed in Sone-Takata-Sugimoto,[(31)] Bobylev-Grzhibovskis-Heinz,[(7)] and Sone-Takata-Golse[(30)] (see also Sone ref. 22).

The half-space problem, linear or nonlinear, plays an important role in the asymptotic behavior of the solution of the boundary-value problem of the

Boltzmann equation for small Knudsen numbers. That is, its solution gives the boundary condition for the fluid-dynamic-type equations derived from the Boltzmann equation and the Knudsen-layer correction to the solution of the fluid-dynamic-type equations in a neighborhood of the boundary. Its detailed discussion is given in refs. 22 and 23. Here, we only mention the role of the result of the nonlinear half-space problem summarized in this subsection. Consider a gas around its condensed phase where evaporation or condensation of a finite Mach number is taking place. The behavior of the gas (or the solution of the Boltzmann equation) in the limit that Knudsen number vanishes is given by the solution of the Euler equations whose boundary condition on the interface is given by (2.3), (2.4), or (2.5) where the values with subscript ∞ are taken as the boundary values of the corresponding variables in the Euler equations.

2.2. Structure of Bifurcation of a Solution

The dimension of the range of existence of a solution in the parameter space $(M_{n\infty},\ M_{t\infty},\ p_\infty/p_w,\ \theta_\infty/\theta_w)$ increases by two[5] on the transition from the evaporating flow to the subsonic condensing flow and increases by one on the transition from the subsonic condensing flow to the supersonic condensing flow. Analytical structure of the bifurcation of a solution is studied in Sone[(20)] for the former transition and in Sone-Golse-Ohwada-Doi[(26)] for the latter (see also Sone refs. 21 and 23). These are studied on the basis of the standard Boltzmann equation as well as the BKW equation.

The transition from evaporation ($M_{n\infty} > 0$) to condensation ($M_{n\infty} < 0$) is studied by a perturbation analysis to the uniform state $F(z, v) = \mathcal{M}_{(\rho_w,0,\theta_w)}$. Putting $F(z, v) = \mathcal{M}_{(\rho_w,0,\theta_w)}(1 + \phi)$, we consider that ϕ is small, i.e., $\phi = O(|M_{n\infty}|)$ with $|M_{n\infty}| \ll 1$. The linearized problem where the second and higher-order terms in $|M_{n\infty}|$ are neglected is studied mathematically (e.g., Bardos-Caflisch-Nicolaenko, ref. 5,[6] Golse-Poupaud, ref. 16; see Sec. 3) and numerically (e.g., ref. 28 for the BKW model, ref. 27 for a hard-sphere gas). The solution of the linearized problem is determined only by the parameter $M_{n\infty}$ irrespective of the sign of $M_{n\infty}$, as will be discussed in the next section. For the original nonlinear Boltzmann equation, the situation is entirely different, however small $|M_{n\infty}|$ may be.

The boundary condition on the condensed phase being put aside, the nonlinear Boltzmann equation has a slowly varying solution whose length scale of variation is $1/|M_{n\infty}|$ for small $|M_{n\infty}|$.[7] The nontrivial leading term of the slowly varying

[5] Three if we count $u_{x\infty}$ and $u_{y\infty}$ independently.

[6] The theorem proved in ref. 5 is conjectured by Grad.[(17)]

[7] (i) Let a characteristic density of the gas be taken unity, i.e., $mn = 1$, where n is the corresponding number density of the molecules of the gas. Then, $nd_m^2/2 = 1$, since $d_m^2/2m$ is taken to be unity just after (1.3) by the choice of the unit length. That is, $2/nd_m^2$, which is of the order of the mean free path,

solution is a *local Maxwellian*. The parameters of the local Maxwellian, i.e., density ρ (or pressure p), flow velocity $u = (u_x, u_y, u_z)$, and temperature θ, are determined by the *incompressible Navier–Stokes* equations with their *energy* equation being a little *modified* (i.e., internal energy or thermal conductivity being multiplied by 5/3 or 3/5).[8] Thus, for the case of condensation ($u_{z\infty} < 0$ or $M_{n\infty} < 0$), there is a non-trivial bounded solution in the half-space owing to the convection effect of flow blowing toward the condensed phase. For a given set of the macroscopic variables at $z = 0$, we can choose the parallel velocity component (or the velocity component parallel to the condensed phase) and temperature at infinity arbitrarily.

In view of the properties of the solution of the linearized problem and the slowly varying solution,[9] we can construct the solution of the condensation problem. That is, the pressure, flow velocity, and temperature at infinity of the linearized problem are taken as those at $z = 0$ of the slowly varying solution. Then, the two solutions are continuously connected at the level of the velocity distribution

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is chosen as the unit of length in the Boltzmann equation (1.2) with (1.3). Thus, the natural length scale of variation of the variables in the half-space problem of the Boltzmann equation (1.2) is the mean free path, since there is no geometric characteristic length. This property applies to the standard Boltzmann equation in general, including the BKW equation, as well as the Boltzmann equation (1.2) with (1.3) for a hard-sphere gas. Thus, the statement that the length scale of variation of a variable is of the order of unity means that the variable makes an appreciable change over the mean free path. When the length scale of variation is of the order of $1/|M_{n\infty}|$, the nonlinear term of the order of ϕ^2 is of the same order as the space-derivative term $\partial_z \phi$, since $\phi = O(|M_{n\infty}|)$ and $\partial_z(*) = O(|M_{n\infty}|*)$. Incidentally, for clearer physical discussion, a systematic system of non-dimensional variables as in ref. 22 or 23 should be introduced.

(ii) The spatial coordinate z being scaled by $1/|M_{n\infty}|$, i.e., $Z = z|M_{n\infty}|$, the collision term $\mathcal{B}(F, F)$ is relatively multiplied by $1/|M_{n\infty}|$ in the Boltzmann equation (1.2) with the new variable Z. Then, the solution is looked for in a power series of $|M_{n\infty}|$ with the additional assumption that the perturbation of F from a uniform state at rest is of the order of $|M_{n\infty}|$. This is a special class of the Hilbert solution[(18)] with $|M_{n\infty}|$ as the Knudsen number and with the perturbation from a uniform state at rest of the order of $|M_{n\infty}|$.[(20,22,23)] The solution thus obtained is the slowly varying solution. Obviously from the above explanation, the nontrivial leading term of the solution is a local Maxwellian. Incidentally, there is a slowly varying solution with small $|M_{n\infty}|$ but with a finite temperature variation, for which the connection analysis similar to that to be explained for the preceding case can be carried out, and the solution with small $|M_{n\infty}|$ but finite $|\theta_\infty/\theta_w - 1|$ of the half-space problem can be constructed.[(21,22)]

[8] The stress is given by an isotropic tensor or pressure and the heat flow vanishes for a Maxwellian. This does not mean that the governing equations for the macroscopic variables are the Euler equations. A little careful examination of the order of the terms in the conservation equations is required in view of the size of the perturbations and the length scale of variation of the variables. Incidentally, the gas is not incompressible, but the flow velocity is determined by the incompressible Navier–Stokes equations for small Mach numbers and small temperature variations. The difference of the energy equation is due to the work done by pressure, which is a compressibility effect. See the discussion in refs. 22 and 23.

[9] In the region where the length scale of variation of the solution is of the order of unity, the solution of the linearized Boltzmann equation is the leading-order solution.

function.[10] The temperature and parallel velocity component at infinity in the resulting solution can be chosen arbitrarily. Thus, we obtain a solution with more freedom than the evaporating solution.

We have seen that existence of a slowly varying solution plays an important role for the jump of the dimension of the range of existence of a solution. Looking for a slowly varying solution around other states, we find it around the point where the normal velocity u_z is sonic, i.e., $|u_z|/(\frac{5}{3}\theta)^{1/2} = 1$.[11] The slowly varying condensing solution ($u_z < 0$) bounded in a half space is of two kinds: one is a weak shock wave solution (Caflisch-Nicolaenko,(8) Grad,(17) Liu-Yu(19)) and the other is a supersonic accelerating flow, in both of which the length scale of variation of the solution is $1/|M_{n\infty} + 1|$, where $|M_{n\infty} + 1|$ is small, and the parallel velocity component is uniform; the flow field with nonzero parallel velocity component is given by parallel translation of the field with zero parallel velocity component. Nontrivial leading terms of these solutions are expressed by local Maxwellians, and the parameters p, u_z, and θ in the Maxwellians are determined by the *compressible Navier–Stokes equations* for small variations of flow velocity and temperature around a sonic condition.[12] The pressure p and temperature θ of these solutions are expressed parametrically with Mach number M_n, defined by $M_n = u_z/(\frac{5}{3}\theta)^{1/2}$, as[13]

$$\frac{p}{p_\infty} = 1 + \frac{5}{4}(M_n - M_{n\infty}),$$
$$\frac{\theta}{\theta_\infty} = 1 + \frac{1}{2}(M_n - M_{n\infty}). \tag{2.7}$$

The slowly varying solutions expressing a flow in the $-z$ direction exist only for $M_{n\infty} < -1$, and M_n ranges in the domain

$$\begin{aligned} &M_{n\infty} \le M_n \le -M_{n\infty} - 2 \quad \text{(weak shock wave)},\\ &M_n \le M_{n\infty} \quad \text{(supersonic accelerating flow)}. \end{aligned} \tag{2.8}$$

[10] The effect of the derivative of slowly varying solution enters the next-order solution.

[11] Take a uniform state, i.e., a uniform Maxwellian (say, F_E). Consider a slowly varying solution whose deviation from F_E is small, i.e., $F = F_E(1 + \varepsilon\varphi)$, where ε is a small parameter and φ is the perturbed solution of the order of unity, and whose length scale of variation is of the order of $1/\varepsilon$, i.e., $\partial\varphi/\partial z = O(\varepsilon\varphi)$. The coordinate z being scaled by $1/\varepsilon$, the solution φ is looked for in a power series of ε by a similar analysis to that explained in (ii) of Footnote 7. A nontrivial solution exists only when F_E is a Maxwellian at rest or at sonic condition. The former is the preceding case with small $|M_{n\infty}|$, for which $\varepsilon = |M_{n\infty}|$. For the latter, $\varepsilon = |M_{n\infty} \pm 1|$ depending on $M_{n\infty} \lessgtr 0$. See refs. 21, 23 or 26 or for the details.

[12] See the first three sentences in Footnote 8.

[13] This is the same as what is obtained in an isentropic flow for small $|M_n - M_{n\infty}|$. This is naturally understood, since the entropy variation through a weak shock wave is of the third order of the strength (or the pressure jump) of the shock wave.

It may be noted that the solutions can be shifted arbitrarily in the z direction and that $M_{n\infty}$ and $-M_{n\infty} -2$ in the weak shock wave are the values of M_n at the upstream and downstream infinities related by the shock condition.

The solution whose length scale of variation with respect to z is of the order of unity[14] is studied in detail numerically. The solution exists on the hypersurface $p_\infty/p_w = F_s(M_{n\infty}, |M_{t\infty}|, \theta_\infty/\theta_w)$ in $(M_{n\infty}, M_{t\infty}, p_\infty/p_w, \theta_\infty/\theta_w)$ space for $-1 < M_{n\infty} < 0$ (Fig. 1), and it approaches its uniform state at infinity exponentially. Further, the same type of solution (i.e., a solution with length scale of variation of the order of unity and exponential approach to the state at infinity) exists on a hypersurface $p_\infty/p_w = F_{es}(M_{n\infty}, |M_{t\infty}|, \theta_\infty/\theta_w)$ for $M_{n\infty} \leq -1$ (Fig. 1), where the function $F_{es}(M_{n\infty}, |M_{t\infty}|, \theta_\infty/\theta_w)$ is a smooth extension of $F_s(M_{n\infty}, |M_{t\infty}|, \theta_\infty/\theta_w)$ and a decreasing function of $M_{n\infty}$.(21,26) We will call these solutions (subsonic or supersonic) Knudsen-layer-type solutions.

The nontrivial leading term of the slowly varying solution being local Maxwellian, the two kinds of solutions, the Knudsen-layer-type solution and the slowly varying solution, are continuously connected at the level of the velocity distribution function by connecting the macroscopic variables, at infinity for the former and at $z = 0$ for the latter,[15] in the Maxwellians of the two solutions. Let us rewrite (2.7) in the following form:

$$\frac{p_\infty}{p_w} = \frac{p}{p_w}\Big[1 - \frac{5}{4}(M_n - M_{n\infty})\Big],$$
$$\frac{\theta_\infty}{\theta_w} = \frac{\theta}{\theta_w}\Big[1 - \frac{1}{2}(M_n - M_{n\infty})\Big], \tag{2.9}$$

where p_w and θ_w are inserted for the convenience of the following explanation. The expression gives the values p_∞ and θ_∞ at infinity with the parameter $M_{n\infty}$ when the variables p, θ, and M_n on a point of a slowly varying solution are given. The range of $M_{n\infty}$ is

$M_n \leq M_{n\infty} < -1$ (M_n on a supersonic accelerating flow),

$M_{n\infty} \leq M_n \leq -1$ (M_n on the supersonic part of a shock wave),

$M_{n\infty} \leq -2 - M_n \leq -1$ (M_n on the subsonic part of a shock wave). (2.10)

Let a Knudsen-layer-type solution be given. Its parameters determined by the variables at infinity are denoted in parentheses with subscript K as $(M_{n\infty})_K$, $(M_{t\infty})_K$, $(p_\infty/p_w)_K$, and $(\theta_\infty/\theta_w)_K$. The $(M_{n\infty})_K$, $(p_\infty/p_w)_K$, and $(\theta_\infty/\theta_w)_K$ are, respectively, chosen as M_n, p/p_w, and θ/θ_w in (2.9).[16] Then, the connection of the

[14] See Footnote 7 (i).

[15] By the freedom of shift of the slowly varying solution mentioned above, any point along the solution can be positioned at $z = 0$.

[16] According to Footnote 15, the state can be put at $z = 0$.

Knudsen-layer-type solution and the slowly varying solution is completed.[17] The connected solution forms a one-parameter family of solutions with the parameter $M_{n\infty}$ in the range (2.10) with $M_n = (M_{n\infty})_K$. It goes without saying that $(M_{n\infty})_K$ and $M_{n\infty}$ (< -1) have to be close to -1.

From the above discussion, the solution exists when the parameters $M_{n\infty}$, $M_{t\infty}$, p_∞/p_w, and θ_∞/θ_w satisfy

$$\frac{p_\infty}{p_w} = \left(\frac{p_\infty}{p_w}\right)_K \left\{1 - \frac{5}{4}[(M_{n\infty})_K - M_{n\infty}]\right\}, \tag{2.11a}$$

$$\frac{\theta_\infty}{\theta_w} = \left(\frac{\theta_\infty}{\theta_w}\right)_K \left\{1 - \frac{1}{2}[(M_{n\infty})_K - M_{n\infty}]\right\}, \tag{2.11b}$$

$$M_{t\infty}(\theta_\infty/\theta_w)^{1/2} = (M_{t\infty})_K((\theta_\infty/\theta_w)_K)^{1/2}, \tag{2.11c}$$

where the relation (2.11c) corresponds to $u_{x\infty} = (u_{x\infty})_K$,[18] and the parameters $(M_{n\infty})_K$, $(M_{t\infty})_K$, $(p_\infty/p_w)_K$, and $(\theta_\infty/\theta_w)_K$ of the Knudsen-layer-type solution are on the hypersurface F_s or F_{es}, i.e.,

$$(p_\infty/p_w)_K = F_S((M_{n\infty})_K, (|M_{t\infty}|)_K, (\theta_\infty/\theta_w)_K), \tag{2.12}$$

with

$$\begin{aligned} F_S &= F_s((M_{n\infty})_K, (|M_{t\infty}|)_K, (\theta_\infty/\theta_w)_K) \ \text{ for } \ (M_{n\infty})_K > -1, \\ &= F_{es}((M_{n\infty})_K, (|M_{t\infty}|)_K, (\theta_\infty/\theta_w)_K) \ \text{ for } \ (M_{n\infty})_K \le -1. \end{aligned} \tag{2.13}$$

We have a free parameter, i.e., $M_{n\infty}$, in addition to $(M_{n\infty})_K$, $(\theta_\infty/\theta_w)_K$, and $(|M_{t\infty}|)_K$ in the formulas (2.11a)–(2.13). That is, the dimension of the range of existence of a solution of the condensing flow increases by one by transition from a subsonic solution to a supersonic one. According to (2.10), the range of the parameter $M_{n\infty}$ is

$$\begin{aligned} &\text{(i) } M_{n\infty} \le -1 \qquad \text{for } (M_{n\infty})_K \le -1, \\ &\text{(i-a) } (M_{n\infty})_K \le M_{n\infty} \le -1 \qquad \text{(accelerating flow)}, \\ &\text{(i-b) } M_{n\infty} \le (M_{n\infty})_K \qquad \text{(weak shock wave)}, \\ &\text{(ii) } M_{n\infty} < -2 - (M_{n\infty})_K < -1 \qquad \text{for } (M_{n\infty})_K > -1. \end{aligned} \tag{2.14}$$

The range of existence of a solution with a supersonic accelerating flow or a supersonic part of a shock wave as its slowly varying part extends from the hypersurface $p_\infty/p_w = F_{es}(M_{n\infty}, |M_{t\infty}|, \theta_\infty/\theta_w)$ in $(M_{n\infty}, M_{t\infty}, p_\infty/p_w, \theta_\infty/\theta_w)$

[17] Naturally, trivial connection $(u_{x\infty})_K = u_x$ (= const.) of the slowly varying solution is made. The choice of u_x does not affect the other variables in the slowly varying solution.
[18] Note that the parallel velocity is uniform in the slowly varying solution.

space. That is, in view of (2.11a) and (i-a,b) in (2.14), the range of existence of a solution with a part of a supersonic accelerating flow extends to the side closer to $M_{n\infty} = -1$ and higher p_∞/p_w, and that of a solution with a supersonic part of a shock wave extends to the side farther from $M_{n\infty} = -1$ and lower p_∞/p_w. In view of (2.11a) and (ii) in (2.14), the range of existence of a solution with a subsonic part (and the full supersonic part) of a weak shock wave extends in $M_{n\infty} < -2 - (M_{n\infty})_K$, i.e., from $M_{n\infty} = -2 - (M_{n\infty})_K$ to the side farther from $M_{n\infty} = -1$ and lower p_∞/p_w. The boundary $M_{n\infty} = -2 - (M_{n\infty})_K$ corresponds to the second equal sign of the relation for a weak shock wave in (2.8), that is, $(M_{n\infty})_K$ corresponds to the downstream infinity of the shock wave. When the connection of the two solutions are made here, the profile or nonuniform part of the weak shock wave shifts upstream infinity and disappears from the flow field. Thus, it is excluded from the range of existence of a solution. A point on the hypersurface $p_\infty/p_w = F_b(M_{n\infty}, |M_{t\infty}|, \theta_\infty/\theta_w)$ is related to a point on $p_\infty/p_w = F_s(M_{n\infty}, |M_{t\infty}|, \theta_\infty/\theta_w)$ by the Rankine-Hugoniot relation. From the way of its determination, it is easily seen that the shape of the boundary F_b is due to the dependence of the speed of propagation of a shock wave (nonlinear wave) on its amplitude or pressure ratio. In the linear theory by Coron-Golse-Sulem[(12)] to be explained in Sec. 3, the boundary is given by $M_{n\infty} = -1$.

The case where $|M_{n\infty} + 1| \ll 1$ and $|(M_{n\infty})_K + 1| \ll 1$ being considered here, the range given by (2.11a)–(2.13) can be simplified. The free parameters $(M_{t\infty})_K$ and $(\theta_\infty/\theta_w)_K$ in (2.12) for $(p_\infty/p_w)_K$ are expressed with $M_{t\infty}$, θ_∞/θ_w, and $(M_{n\infty})_K - M_{n\infty}$ with the aid of (2.11b) and (2.11c). These relations being substituted into (2.11a), p_∞/p_w is expressed with $M_{n\infty}$, $|M_{t\infty}|$, θ_∞/θ_w, and $(M_{n\infty})_K$.[19] Neglecting the second and higher-order terms of $|M_{n\infty} + 1|$ and $|(M_{n\infty})_K + 1|$ in this expression, we obtain the following formula for the range of existence of a supersonic condensing flow:[20]

$$\frac{p_\infty}{p_w} = F_S + \left(\frac{5}{2}F_S - \frac{\partial F_S}{\partial a_1} + \frac{|M_{t\infty}|}{2}\frac{\partial F_S}{\partial a_2} - \frac{\theta_\infty}{\theta_w}\frac{\partial F_S}{\partial a_3}\right)(M_{n\infty} + 1)$$
$$+ \left(\frac{5}{4}F_S - \frac{\partial F_S}{\partial a_1} + \frac{|M_{t\infty}|}{4}\frac{\partial F_S}{\partial a_2} - \frac{\theta_\infty}{2\theta_w}\frac{\partial F_S}{\partial a_3}\right)t,$$
$$t = -(M_{n\infty})_K - M_{n\infty} - 2 > 0,\ M_{n\infty} < -1, \tag{2.15}$$

where t is a positive free parameter, $\partial F_S/\partial a_i$ is the partial derivative with respect to the i-th argument of F_S, and the three arguments of F_S and $\partial F_S/\partial a_i$'s are, commonly, -1, $|M_{t\infty}|$, and θ_∞/θ_w in their order, e.g., $F_S(-1, |M_{t\infty}|, \theta_\infty/\theta_w)$. If the sign of the quantity in the parentheses of the last term is positive, which

[19] In addition to $(M_{n\infty})_K - M_{n\infty}$ in the relations, the variable $(M_{n\infty})_K$ is contained in F_S in (2.12).

[20] In this analysis, it is assumed that F_s and F_{es} are smoothly connected, i.e., F_S is smooth, which is confirmed for the BKW model numerically.

is confirmed for the BKW model numerically, the range is given by $p_\infty/p_w > F_S + (\cdots)(M_{n\infty}+1)$, as shown in Fig. 1. The boundary function F_b of the range is given by

$$F_b = F_S + \left(\frac{5}{2}F_S - \frac{\partial F_S}{\partial a_1} + \frac{|M_{t\infty}|}{2}\frac{\partial F_S}{\partial a_2} - \frac{\theta_\infty}{\theta_w}\frac{\partial F_S}{\partial a_3}\right)(M_{n\infty}+1), \qquad (2.16)$$

where the arguments of F_S and $\partial F_S/\partial a_i$ are the same as in (2.15).

2.3. Supplementary Notes

The above results are for the complete condensation boundary condition. The results are extended to a more general boundary condition given by

$$F(z, v) = \mathcal{M}_{(\alpha_c\rho_w+(1-\alpha_c)\sigma_w,0,\theta_w)} \quad (v_z > 0) \quad \text{at } z = 0,$$

$$\sigma_w = -\left(\frac{2\pi}{\theta_w}\right)^{1/2}\int_{v_z<0} v_z F(0, v)dv,$$

where α_c is a constant ($0 < \alpha_c \leq 1$) called the condensation coefficient (see refs. 22 and 23). When $M_{n\infty} \geq 0$ (an evaporating flow), the character of the solution is the same as that for the complete condensation condition, that is, the solution is determined by one parameter, e.g., $M_{n\infty}$. When $M_{n\infty} < 0$ (a condensing flow), the character of the solution is subject to some change. The diagram $p_\infty/p_w = F_s(M_{n\infty}, |M_{t\infty}|, \theta_\infty/\theta_w)$ and $p_\infty/p_w > F_b(M_{n\infty}, |M_{t\infty}|, \theta_\infty/\theta_w)$ remains qualitatively unchanged when the condensation coefficient α_c (≤ 1) is larger than some value ($\alpha_c > \alpha_c^{cr}$). For $\alpha_c \leq \alpha_c^{cr}$, $F_s \to \infty$ as $M_{n\infty} \to -c_{1+}$ ($c_1 \leq 1$) and $F_b \to \infty$ as $M_{n\infty} \to -c_{2-}$ ($c_2 \geq 1$), where c_1 and c_2 are determined by α_c, θ_∞/θ_w, and $|M_{t\infty}|$. There is a band region $c_1 \leq -M_{n\infty} \leq c_2$ where no solution exists when $\alpha_c \leq \alpha_c^{cr}$ (see Fig. 2).[21]

A similar result is obtained when there is some non-condensable gas in the gas (Sone-Aoki-Doi[(24)]). In the evaporating flow, the non-condensable component is blown off up to infinity and disappears from the flow field. In the condensing flow, the diagram of the range of existence of a solution is qualitatively similar to that for $\alpha_c > \alpha_c^{cr}$ (or without a non-condensable gas) when the amount of the non-condensable gas is smaller than some value; and the diagram is similar to that

[21] The above result is derived under the assumptions on F_s and F_b for the complete condensation condition that $-M_{n\infty}F_s$ is a decreasing function and $-M_{n\infty}F_b$ is an increasing function with respect to $M_{n\infty}$ in $M_{n\infty} < 0$ and that the relation (2.6) holds, which are confirmed for the BKW model numerically. Then, $F_s(M_{n\infty} = -1_+)$ and $F_b(M_{n\infty} = -1_-)$ commonly diverge to $+\infty$ as $\alpha_c \to \alpha_{c+}^{cr}$. Thus, $c_1 = c_2 = 1$ at $\alpha_c = \alpha_c^{cr}$ and no solution exists at $M_{n\infty} = -1$. Even without the above assumptions, there appears a band region or regions where no solution exists in both the subsonic and supersonic regions as α_c decreases.

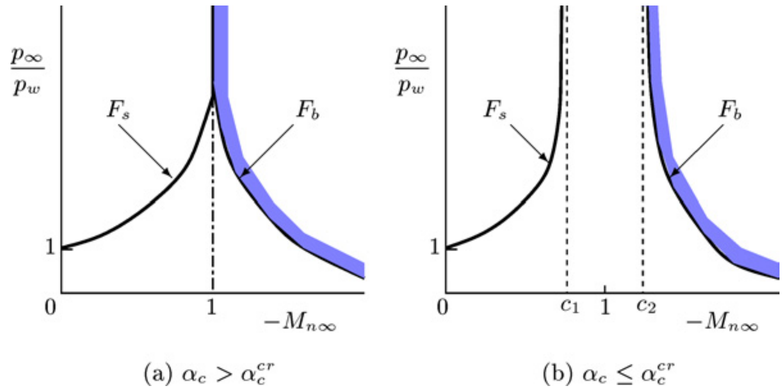

Fig. 2. The diagram of the range of existence of the solution of a condensing flow for a generalized boundary condition. (a) $\alpha_c > \alpha_c^{cr}$ and (b) $\alpha_c \leq \alpha_c^{cr}$. The dashed lines in panel (b) are the asymptotes of F_s and F_b. No solution exists in $c_1 \leq -M_{n\infty} \leq c_2$ when $\alpha_c \leq \alpha_c^{cr}$.

for $\alpha_c \leq \alpha_c^{cr}$, i.e., there is a band region without a solution around $M_{n\infty} = -1$, when the amount is larger than that value.

3. THE LINEARIZED THEORY

In this section, we choose units of pressure and temperature so that $p_\infty = 1$ and $\theta_\infty = 1$, so that the condition at infinity (1.6) becomes

$$F(z, v) \to \mathcal{M}_{(1,u_\infty,1)}(v) \text{ as } z \to +\infty\,. \tag{3.1}$$

Next, we assume that the deviation of the phase-space density from the Maxwellian at infinity is so small that it becomes legitimate to replace (1.2) with its linearization at the Maxwellian state at infinity. Also, we change the velocity variable v into ξ defined by

$$\xi = v - u_\infty \tag{3.2}$$

and we normalize the linearization of the Boltzmann collision integral by setting

$$F(z, v) = \mathcal{M}_{(1,0,1)}(v - u_\infty)(1 + f(z, v - u_\infty))\,.$$

Furthermore, we henceforth use the short-hand notation

$$M(\xi) = \mathcal{M}_{(1,0,1)}(\xi)\,.$$

The linearization of (1.2) at the Maxwellian state at infinity in these new variables becomes

$$(\xi_z + u_{z\infty})\partial_z f + \mathcal{L}f = 0\,, \quad z > 0\,,\ \xi \in \mathbf{R}^3\,, \tag{3.3}$$

where $\mathcal{L}$ is the linearization at M of the Boltzmann collision integral, defined as

$$\mathcal{L}f(v) = \iint_{\mathbf{R}^3\times\mathbf{S}^2} (f + f_* - f' - f'_*)|(v - v_*)\cdot\omega| M_* dv_* d\omega. \tag{3.4}$$

The condition at infinity (1.6) becomes

$$f(z,\xi) \to 0 \text{ as } z \to +\infty, \quad \xi \in \mathbf{R}^3. \tag{3.5}$$

We shall restrict our attention to the linearization at M of the class of boundary conditions at $z = 0$ considered in the previous section, i.e.

$$f(0,\xi) = f_b(\xi), \quad \xi_z + u_{z\infty} > 0. \tag{3.6}$$

For instance, consistently with the discussion in the previous section, one could consider as boundary data

$$f_b(\xi) = \frac{1}{M(\xi)}\left(\mathcal{M}_{(\rho_w,-u_\infty,\theta_w)}(\xi) - M(\xi)\right),$$

or the linearization thereof

$$f_b(\xi) = (\rho_w - 1) - u_\infty\cdot\xi + (\theta_w - 1)\frac{1}{2}(|\xi|^2 - 3).$$

Before going further, we recall some basic facts on the linearized operator $\mathcal{L}$, that are due to Hilbert.[18] The linearized collision integral $\mathcal{L}$ is an unbounded, self-adjoint Fredholm operator on $L^2(\mathbf{R}^3, Md\xi)$, with domain

$$D(\mathcal{L}) = L^2(\mathbf{R}^3; (1 + |\xi|^2)Md\xi),$$

and nullspace

$$\ker\mathcal{L} = \operatorname{span}\{1, \xi_x, \xi_y, \xi_z, |\xi|^2\}.$$

Let Π be the orthogonal projection on $\ker\mathcal{L}$ in $L^2(\mathbf{R}^3; Md\xi)$, Hilbert's results imply that one has a spectral gap estimate for $\mathcal{L}$, i.e. there exists a positive constant c_0 such that, for each $\phi \in D(\mathcal{L})$

$$\int_{\mathbf{R}^3} \phi\mathcal{L}\phi M d\xi \geq c_0 \int_{\mathbf{R}^3} (\phi - \Pi\phi)^2 M d\xi.$$

(Recently, Baranger-Mouhot proved this inequality by a constructive argument that avoids Hilbert's compactness method and provides an explicit formula for c_0). An elementary—but important—observation by Bardos-Caflisch-Nicolaenko in ref. 5 says that the spectral gap estimate above can be strengthened into

$$\int_{\mathbf{R}^3} \phi\mathcal{L}\phi M d\xi \geq c_0 \int_{\mathbf{R}^3} (\phi - \Pi\phi)^2\,(1 + |\xi|)M d\xi. \tag{3.7}$$

Bardos-Caflisch-Nicolaenko studied the case $u_{z\infty} = 0$ in ref. 5 and proved the following result.

Theorem 3.1. *Let $f_b \equiv f_b(\xi)$ be any measurable function such that*

$$\int_{\xi_z>0} f_b(\xi)^2(1 + |\xi|)M(\xi)d\xi < +\infty .$$

Then, the problem (3.3)–(3.6) has a unique solution $f \equiv f(z, \xi)$ that belongs to $L^\infty(\mathbf{R}_+; L^2(\mathbf{R}^3, (1 + |\xi|)Md\xi))$ and has zero mass-flux:

$$\int_{\mathbf{R}^3} \xi_z f(z, \xi)Md\xi = 0 . \tag{3.8}$$

Furthermore, this solution has the following asymptotic behavior at infinity: there exists a unique $q_\infty \in span\{1, \xi_x, \xi_y, |\xi|^2\}$ such that

$$e^{\gamma z}(f(z, \xi) - q_\infty(\xi)) \in L^\infty(\mathbf{R}_+; L^2(\mathbf{R}^3, (1 + |\xi|)Md\xi)).$$

for all small enough $\gamma > 0$.

For instance, if

$$f_b(\xi) = (\rho_w - 1) - u_\infty \cdot \xi + (\theta_w - 1)\frac{1}{2}(|\xi|^2 - 3),$$

then the function

$$g(z, \xi) = (\rho_w - 1) - u_\infty \cdot \xi + (\theta_w - 1)\frac{1}{2}(|\xi|^2 - 3)$$

is a trivial solution of (3.3) that satisfies the boundary condition (3.6). However, it does not satisfy the condition (3.8). Instead, the theorem above gives the existence of a function f that satisfies (3.3)–(3.8), and converges as $z \to +\infty$ to an infinitesimal Maxwellian of the form

$$q_\infty(\xi) = a_\infty + c_\infty\frac{1}{2}(|\xi|^2 - 3),$$

where the numbers a_∞ and c_∞ are (nonexplicit) linear functionals of the boundary data f_b.

The case of an arbitrary $u_{z\infty}$ has a richer structure, that was conjectured by Cercignani.(10)

Following Bardos-Caflisch-Nicolaenko,(10) we consider the quantity

$$Q[u_{z\infty}](\phi) = \int_{\mathbf{R}^3} (\xi_z + u_{z\infty})\,\phi\,(\xi)^2 M(\xi)\,d\xi. \tag{3.9}$$

This quantity is the second order term in the expansion of the flux of Boltzmann's H function

$$\int_{\mathbf{R}^3} (\xi_z + u_{z\infty}) F(\xi) \ln F(\xi)\, d\xi$$

near $F = M$. More precisely, setting $F = M(1 + f)$, one has

$$\begin{aligned}\int_{\mathbf{R}^3} (\xi_z + u_{z\infty}) F(\xi) \ln F(\xi)\, d\xi &= \int_{\mathbf{R}^3} (\xi_z + u_{z\infty})\, M(\xi)\, \ln M(\xi)\, d\xi \\ &\quad + \int_{\mathbf{R}^3} (\xi_z + u_{z\infty})(\ln M(\xi) + 1) f(\xi) M(\xi)\, d\xi \\ &\quad + \frac{1}{2} Q[u_{z\infty}](f) + O(f^3).\end{aligned}$$

Since $\ker \mathcal{L} = \operatorname{span}\{1, \xi_x, \xi_y, \xi_z, |\xi|^2\}$ and $\mathcal{L}$ is a self-adjoint operator on $L^2((\mathbf{R}^3; M dv)$, one has

$$\frac{d}{dz} \int_{\mathbf{R}^3} (\xi_z + u_{z\infty}) \begin{pmatrix} 1 \\ \xi_x \\ \xi_y \\ \xi_z \\ |\xi|^2 \end{pmatrix} f(z, \xi) M(\xi)\, d\xi = 0. \tag{3.10}$$

Since $\ln M(\xi) + 1 \in \operatorname{span}\{1, \xi_x, \xi_y, \xi_z, |\xi|^2\}$, the relations (3.10) entail

$$\frac{d}{dz} \int_{\mathbf{R}^3} (\xi_z + u_{z\infty})(\ln M(\xi) + 1) f(z, \xi) M(\xi)\, d\xi = 0.$$

Since $\mathcal{L}$ is a nonnegative operator on $L^2((\mathbf{R}^3; M dv)$, one has

$$\frac{d}{dz} Q[u_{z\infty}](f) = -\int_{\mathbf{R}^3} f(z, \xi) \mathcal{L} f(z, \xi) M(\xi)\, d\xi \le 0. \tag{3.11}$$

That $Q[u_{z\infty}](f)$ is nonincreasing (3.11) clearly has interesting implications if we know a priori that this quantity is bounded from below. In this case, $Q[u_{z\infty}](f)$ is a Lyapunov function for the linearized half-space problem under consideration, and has important implications on its stability. Since $Q[u_{z\infty}]$ is a quadratic form, knowing that it is bounded from below is equivalent to knowing that it is a priori nonnegative. Furthermore, we expect that $f(z, \cdot)$ approaches $\ker \mathcal{L}$ as $z \to +\infty$, so that it is enough to know a priori that the restriction of $Q[u_{z\infty}]$ to the finite dimensional subspace $\ker \mathcal{L}$ is bounded from below. On the other hand, the invariance relations (3.10) constrain the projection of f on $\ker \mathcal{L}$ as $z \to +\infty$.

A complete classification of the linearized half-space problem (3.3) was conjectured by Cercignani in ref. 10, and eventually proved by Coron-Golse-Sulem in ref. 12. Their result is summarized in the following.

Theorem 3.2. *For each $u_{z\infty} \in \mathbf{R}$, call*

$$\mathbf{V}^+[u_{z\infty}] = \{V \text{ linear subspace of } \ker\mathcal{L} \mid Q[u_{z\infty}]\big|_V \geq 0\}.$$

Let $\mathcal{V}$ be a maximal element of $\mathbf{V}^+[u_{z\infty}]$ and pick an arbitrary $l \in \ker\mathcal{L}$.

For each measurable boundary data $f_b \equiv f_b(\xi)$ such that

$$\int_{\xi_z>0} f_b(\xi)^2(1+|\xi|)M(\xi)\,d\xi < +\infty,$$

there exists a unique $q \in \mathcal{V}$ and a unique solution f of the problem (3.3)–(3.6) in $L^\infty(\mathbf{R}_+; L^2(\mathbf{R}^3, Md\xi))$ with the following asymptotic behavior as $z \to +\infty$:

$$e^{\gamma z}(f(z,\xi) - q(\xi) - l(\xi)) \in L^\infty(\mathbf{R}_+; L^2(\mathbf{R}^3, (1+|\xi|)Md\xi))$$

for each small enough $\gamma > 0$.

Moreover, this solution satisfies

$$\int_{\mathbf{R}^3} (\xi_z + u_{z\infty}) f(z,\xi)\phi(\xi)M(\xi)\,d\xi = 0, \quad z > 0,$$

for each ϕ in the radical[22] *of $Q[u_{z\infty}]$.*

The following basis of $\ker\mathcal{L}$ is orthonormal for the $L^2(\mathbf{R}^3; Md\xi)$ inner product and orthogonal for the quadratic form $Q[u_{z\infty}]$:

$$\chi_0(\xi) = \frac{1}{\sqrt{30}}(|\xi|^2 + \sqrt{15}\xi_z),$$

$$\chi_1(\xi) = \xi_x, \qquad \chi_2(\xi) = \xi_y,$$

$$\chi_3(\xi) = \frac{1}{\sqrt{10}}(|\xi|^2 - 5),$$

$$\chi_4(\xi) = \frac{1}{\sqrt{30}}(|\xi|^2 - \sqrt{15}\xi_z).$$

Denoting by c the speed of sound defined by the Maxwellian state $\mathcal{M}_{(1,0,1)}$, i.e.

$$c = \sqrt{\frac{5}{3}},$$

one has

$$Q[u_{z\infty}](\chi_0) = u_{z\infty} + c,$$

[22] I.e. for each $\phi \in \ker\mathcal{L}$ satisfying

$$\int_{\mathbf{R}^3} (\xi_z + u_{z\infty})\phi(\xi)\psi(\xi)M(\xi)\,d\xi = 0 \text{ for all } \psi \in \ker\mathcal{L}.$$

$$Q[u_{z\infty}](\chi_1) = Q[u_{z\infty}](\chi_2) = Q[u_{z\infty}](\chi_3) = u_{z\infty},$$
$$Q[u_{z\infty}](\chi_4) = u_{z\infty} - c.$$

Below, we denote $\mathcal{V}^+[u_{z\infty}]$ (resp. $\mathcal{V}^-[u_{z\infty}]$) a maximal subspace of $\ker\mathcal{L}$ such that $Q[u_{z\infty}]\big|_{\mathcal{V}^+[u_{z\infty}]}$ is positive definite (resp. $Q[u_{z\infty}]\big|_{\mathcal{V}^-[u_{z\infty}]}$ is negative definite), and we call $\mathcal{V}^0[u_{z\infty}]$ the radical of $Q[u_{z\infty}]$.

Then one can choose as follows

- if $u_{z\infty} > c$, then
$$\mathcal{V}^+[u_{z\infty}] = \ker\mathcal{L}, \quad \mathcal{V}^0[u_{z\infty}] = \mathcal{V}^-[u_{z\infty}] = \{0\},$$
- if $u_{z\infty} = c$, then
$$\mathcal{V}^+[u_{z\infty}] = \operatorname{span}\{\chi_0, \chi_1, \chi_2, \chi_3\},$$
$$\mathcal{V}^0[u_{z\infty}] = \operatorname{span}\{\chi_4\},$$
$$\mathcal{V}^-[u_{z\infty}] = \{0\},$$
- if $0 < u_{z\infty} < c$, then
$$\mathcal{V}^+[u_{z\infty}] = \operatorname{span}\{\chi_0, \chi_1, \chi_2, \chi_3\},$$
$$\mathcal{V}^0[u_{z\infty}] = \{0\},$$
$$\mathcal{V}^-[u_{z\infty}] = \operatorname{span}\{\chi_4\},$$
- if $u_{z\infty} = 0$, then
$$\mathcal{V}^+[u_{z\infty}] = \operatorname{span}\{\chi_0\},$$
$$\mathcal{V}^0[u_{z\infty}] = \operatorname{span}\{\chi_1, \chi_2, \chi_3\},$$
$$\mathcal{V}^-[u_{z\infty}] = \operatorname{span}\{\chi_4\},$$
- if $-c < u_{z\infty} < 0$, then
$$\mathcal{V}^+[u_{z\infty}] = \operatorname{span}\{\chi_0\},$$
$$\mathcal{V}^0[u_{z\infty}] = \{0\},$$
$$\mathcal{V}^-[u_{z\infty}] = \operatorname{span}\{\chi_1, \chi_2, \chi_3, \chi_4\},$$
- if $u_{z\infty} = -c$, then
$$\mathcal{V}^+[u_{z\infty}] = \{0\},$$
$$\mathcal{V}^0[u_{z\infty}] = \operatorname{span}\{\chi_0\},$$
$$\mathcal{V}^-[u_{z\infty}] = \operatorname{span}\{\chi_1, \chi_2, \chi_3, \chi_4\},$$

- if $u_{z\infty} < -c$, then

$$\mathcal{V}^+[u_{z\infty}] = \mathcal{V}^0[u_{z\infty}] = \{0\}, \quad \mathcal{V}^-[u_{z\infty}] = \ker \mathcal{L}.$$

In the case $u_{z\infty} = 0$: both

$$\mathcal{W}^+ = \text{span}\{\chi_0 + \chi_4, \chi_1, \chi_2, \chi_3\} = \text{span}\{1, \xi_x, \xi_y, |\xi|^2\} \text{ and}$$
$$\mathcal{W}^- = \text{span}\{\chi_0 - \chi_4, \chi_1, \chi_2, \chi_3\} = \text{span}\{\xi_x, \xi_y, \xi_z, |\xi|^2 - 5\}$$

belong to $\mathbb{V}^+[0]$. The result proved by Bardos-Caflisch-Nicolaenko (i.e. Theorem 3.1) is the particular case of Theorem 3.2 with $\mathcal{V} = \mathcal{W}^+$.

4. THE WEAKLY NONLINEAR THEORY

In this section, we return to the nonlinear problem stated in the introduction and discussed from the numerical viewpoint in Sec. 2.

The numerical experiments in Sec. 2 and the analysis in Sec. 3 show the importance of the Maxwellian state at infinity in the statement of the half-space problem. This suggests the following formulation of this problem:

"Given the Maxwellian state $\mathcal{M}_{(\rho_\infty, u_\infty, \theta_\infty)}$ at infinity, to find all boundary data $F_b \equiv F_b(v)$ such that the half-space problem

$$v_z \partial_z F = \mathcal{B}(F, F), \quad z > 0, \ v \in \mathbf{R}^3, \tag{4.1}$$

with boundary condition

$$F(v) = F_b(v), \quad v_z > 0 \tag{4.2}$$

has a unique solution $F \equiv F(z, v)$ such that $F(z, v) \to \mathcal{M}_{(\rho_\infty, u_\infty, \theta_\infty)}$ as $z \to +\infty$."

In this and the next section, all solutions of the Boltzmann equation considered are bounded in z, rapidly decaying in v. In particular, for $v_z \neq 0$, $z \mapsto F(z, v)$ is of class C^1 on $\mathbf{R}_+$.

The set of all boundary data F_b satisfying the conditions above is denoted by $\mathbf{S}[\mathcal{M}_{(\rho_\infty, u_\infty, \theta_\infty)}]$—and can be viewed as the stable manifold of $\mathcal{M}_{(\rho_\infty, u_\infty, \theta_\infty)}$.

As explained above, one can always choose units of temperature and pressure so that $\rho_\infty = \theta_\infty = 1$, in which case our problem reduces to finding $\mathbf{S}[\mathcal{M}_{(1, u_\infty, 1)}]$.

Using a clever variant of the analysis of the linearized problem in ref. 12, Ukai-Yang-Yu[32] arrived at a following, very satisfying local description of $\mathbf{S}[\mathcal{M}_{(1, u_\infty, 1)}]$. We recall that $c = \sqrt{5/3}$ — the speed of sound in an ideal gas at temperature 1.

Theorem 4.1. *Assume that* $u_{z\infty} \notin \{0, \pm c\}$. *Then* $\mathbf{S}[\mathcal{M}_{(1, u_\infty, 1)}]$ *is locally (near* $\mathcal{M}_{(1, u_\infty, 1)}$*) a* C^1 *manifold of codimension* $\dim \mathcal{V}^+[u_{z\infty}]$*—with the notation of Sec. 3.*

That the statement above holds in a neighborhood of $\mathcal{M}_{(1,u_\infty,1)}$ is to be understood as follows: there exists $\epsilon > 0$ small enough such that

$$|F_b(v) - \mathcal{M}_{(1,u_\infty,1)(v)}| \leq \epsilon(1 + |v|)^{-\beta}\mathcal{M}_{(1,u_\infty,1)}(v)^{1/2}$$

for some $\beta > \frac{3}{2}$. In which case, the problem (4.1) with boundary condition (4.2) has a unique solution such that, for $\gamma > 0$ small enough, there exists $C > 0$ for which the following estimate holds for each $z > 0$ and $v \in \mathbf{R}^3$:

$$|F(z, v) - \mathcal{M}_{(1,u_\infty,1)}(v)| \leq Ce^{-\gamma z}(1 + |v|)^{-\beta}\mathcal{M}_{(1,u_\infty,1)}(v)^{1/2}.$$

A few remarks on the result above are in order.

To begin with, Theorem 4.1 is a local result only, and in particular the size of the neighborhood of $\mathcal{M}_{(1,u_\infty,1)}$ considered in that statement may depend in u_∞. For instance, it may fail to be uniformly positive as $u_{z\infty} \to 0$. In particular, it might vanish as $u_{z\infty} \to 0$ so as to prevent any Maxwellian state of the form $\mathcal{M}_{(\rho_w,0,\theta_w)}$ to be in that neighborhood. For that reason, Theorem 4.1 may fail to justify the transition between evaporation and condensation as explained in Sec. 2. However, it is consistent with the results obtained in that section, at least as far as the codimension of the stable manifold is concerned.

That the analysis in ref. 32 avoids the cases $u_{z\infty} \in \{0, \pm c\}$ is only a minor technical problem—see ref. 15 for the treatment of these missing cases.

Finally, the proof of Theorem 4.1 is based on a perturbation analysis and Picard's fixed point theorem. As is well known, this method may lead to solutions of the Boltzmann equation that fail to be nonnegative and therefore lose physical meaning. In a more recent paper,[(33)] the same authors proved the nonlinear stability of that solution in the case $u_{z\infty} < -c$ (supersonic condensation). In other words, the solution of the steady half-space problem (4.1) with boundary condition (4.2) provided by Theorem 4.1 is the long time limit of the solution of a time-dependent half-space problem, known to be everywhere nonnegative. Hence the solution of the steady problem is itself everywhere nonnegative.

5. A UNIQUENESS RESULT IN THE LARGE

The analysis in ref. 32 excludes the particular case $u_{z\infty} = 0$. By a completely different energy method, based on Boltzmann's H Theorem, one arrives a global description of the stable manifold of the Maxwellian $\mathcal{M}_{(1,u_\infty,1)}$ for $u_{z\infty} = 0$. Let us introduce another piece of notation: in the sequel, $\mathcal{S}[u_\infty]$ designates the set

$$\mathcal{S}[u_\infty] = \mathbf{S}[\mathcal{M}_{(1\infty,u_\infty,1\infty)}] \cap \left\{\mathcal{M}_{(\rho_w,0,\theta_w)}\big|_{v_z>0} \mid \rho_w, \theta_w > 0\right\}.$$

Here, $\mathbf{S}[\mathcal{M}_{(\rho_\infty,u_\infty,\theta_\infty)}]$ is the stable manifold of the Maxwellian state $\mathcal{M}_{(\rho_\infty,u_\infty,\theta_\infty)}$. Specifically, $\mathbf{S}[\mathcal{M}_{(\rho_\infty,u_\infty,\theta_\infty)}]$ is the set of boundary data $F_b \equiv F_b(v) \geq 0$ a.e. on $\{v \in \mathbf{R}^3 \mid v_z > 0\}$ such that the half-space problem (1.2) has a solution F that satisfies

$$F(0,v) = F_b(v) \text{ for all } v \in \mathbf{R}^3 \text{ such that } v_z > 0,$$
$$\text{and } F(z,v) \to \mathcal{M}_{(\rho_\infty,u_\infty,\theta_\infty)} \text{ as } z \to +\infty.$$

The solutions of the Boltzmann equation considered in this section are supposed to be bounded in z, and rapidly decaying in v as in the previous section. In addition, we also restrict our attention to positive solutions F such that $\ln F$ has at most polynomial growth as $|v| \to +\infty$, so that Boltzmann's H Theorem holds true for this class of solutions.

Theorem 5.1. *One has*

$$\mathcal{S}[0] = \left\{ \mathcal{M}_{(1,0,1)}\big|_{v_z>0} \right\}.$$

In addition, the only solution F of (1.2) such that

$$F\big|_{z=0,\ v_z>0} = \mathcal{M}_{(1,0,1)} \text{ and } \lim_{z\to+\infty} F(z,v) = \mathcal{M}_{(1,0,1)}$$

is the uniform[23] *Maxwellian state $F \equiv \mathcal{M}_{(1,0,1)}$.*

This theorem is a nonlinear analogue of the uniqueness result in ref. 5; it confirms the numerical results described in Sec. 2. Notice that this result does not require the interaction potential to be that of a hard-sphere gas.

Proof of Theorem 5.1. Let F satisfy

$$\begin{aligned} v_z \partial_z F(z,v) &= \mathcal{B}(F,F)(x,v), \quad v \in \mathbf{R}^3,\ z > 0, \\ F(0,v) &= \mathcal{M}_{(\rho_w,0,\theta_w)}(v), \quad v \in \mathbf{R}^3,\ v_z > 0, \\ F(z,v) &\to \mathcal{M}_{(1,0,1)}(v), \quad \text{as } z \to +\infty. \end{aligned}$$

Multiplying the Boltzmann equation by the collision invariants 1, v_x, v_y, and $|v|^2$, one obtains after integrating in v:

$$\int v_z F dv = \int v_z v_x F dv = \int v_z v_y F dv = \int v_z |v|^2 F dv = 0. \tag{5.1}$$

Next we seek to apply Boltzmann's H Theorem. For $\rho > 0, \theta > 0$ and $u \in \mathbf{R}^3$, set

$$\mathcal{F}(F|\mathcal{M}_{(\rho,u,\theta)}) = \int v_z \left[F \ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right) - F + \mathcal{M}_{(\rho,u,\theta)} \right] dv.$$

Because of the relations (5.1) on the fluxes of conserved quantities,

$$\mathcal{F}(F|\mathcal{M}_{(1,0,1)}) = \int v_z F \ln F dv.$$

[23] I.e. constant in z.

Hence Boltzmann's H Theorem reads

$$\frac{d}{dz}\mathcal{F}(F|\mathcal{M}_{(1,0,1)}) - \int \mathcal{B}(F,F)\ln F dv = 0$$

so that, after integrating in z over the half-line $(0, +\infty)$:

$$\lim_{z\to+\infty}\mathcal{F}(F|\mathcal{M}_{(1,0,1)}) - \int_0^{+\infty}\int \mathcal{B}(F,F)\ln F dvdz = \mathcal{F}(F|_{z=0}|\mathcal{M}_{(1,0,1)}).$$

Since the first term on the left-hand side of the equality above is 0, one arrives at the inequality

$$\begin{aligned} 0 &\le -\int_0^{+\infty}\int \mathcal{B}(F,F)\ln F dvdz \\ &= \int_{v_z>0}\left[\mathcal{M}_{(\rho_w,0,\theta_w)}\ln\left(\frac{\mathcal{M}_{(\rho_w,0,\theta_w)}}{\mathcal{M}_{(1,0,1)}}\right) - \mathcal{M}_{(\rho_w,0,\theta_w)} + \mathcal{M}_{(1,0,1)}\right] v_z dv \\ &\quad - \int_{v_z<0}\left[F(0,v)\ln\left(\frac{F(0,v)}{\mathcal{M}_{(1,0,1)}(v)}\right) - F(0,v) + \mathcal{M}_{(1,0,1)}(v)\right]|v_z|dv \end{aligned}$$

At this point, we use the following lemma which is a variant of the Darrozes-Guiraud inequality (see ref. 14).

Lemma 5.2. *Let $\Phi \equiv \Phi(v)$ be defined for all v's in $\mathbf{R}^3$ such that $v_z > 0$. Let $M_{(\rho,0,\theta)}$ be the Maxwellian state satisfying the relations*

$$\int_{v_z>0} v_z \begin{pmatrix} 1 \\ v_y \\ v_z \\ |v|^2 \end{pmatrix} \mathcal{M}_{(\rho,0,\theta)}(v)dv = \int_{v_z>0} v_z \begin{pmatrix} 1 \\ v_y \\ v_z \\ |v|^2 \end{pmatrix} \Phi(v)\, dv \qquad (5.2)$$

Then

$$\begin{aligned} &\int_{v_z>0}\left[\Phi\ln\left(\frac{\Phi}{\mathcal{M}_{(1,0,1)}}\right) - \Phi + \mathcal{M}_{(1,0,1)}\right] v_z dv \\ &\quad - \int_{v_z>0}\left[\mathcal{M}_{(\rho,0,\theta)}\ln\left(\frac{\mathcal{M}_{(\rho,0,\theta)}}{\mathcal{M}_{(1,0,1)}}\right) - \mathcal{M}_{(\rho,0,\theta)} + \mathcal{M}_{(1,0,1)}\right] v_z dv \\ &\quad = \int_{v_z>0}\left[\Phi\ln\left(\frac{\Phi}{\mathcal{M}_{(\rho,0,\theta)}}\right) - \Phi + \mathcal{M}_{(\rho,0,\theta)}\right] v_z dv \ge 0. \end{aligned}$$

Taking this lemma for granted, apply it to $\Phi(v) = F(0, v_x, v_y, -v_z)$. The relations (5.2) with $\rho = \rho_w$ and $\theta = \theta_w$ coincide with the flux conditions (5.1), and hence

$$\begin{aligned} 0 &\leq -\int_0^{+\infty}\int \mathcal{B}(F,F)\ln F\,dv\,dz \\ &= -\int_{v_z<0}\left[F(0,v)\ln\left(\frac{F(0,v)}{\mathcal{M}_{(\rho_w,0,\theta_w)}}\right) - F(0,v) + \mathcal{M}_{(\rho_w,0,\theta_w)}\right]|v_z|\,dv \leq 0 \end{aligned}$$

Therefore

$$-\int_0^{+\infty}\int \mathcal{B}(F,F)\ln F\,dv\,dz = 0, \qquad \text{and}$$

$$\int_{v_z<0}\left[F(0,v)\ln\left(\frac{F(0,v)}{\mathcal{M}_{(\rho_w,0,\theta_w)}}\right) - F(0,v) + \mathcal{M}_{(\rho_w,0,\theta_w)}\right]|v_z|\,dv = 0.$$

The first relation implies that F is everywhere a local Maxwellian, and thus $\mathcal{B}(F,F) = 0$ on $(0,+\infty)\times\mathbf{R}^3$. Boltzmann's equation then implies that $F(z,v)$ is constant in z for each $v\in\mathbf{R}^3$ s.t. $v_z\neq 0$. Thus one has $F(z,v) = \mathcal{M}_{(1,0,1)}(v)$ for all $z\geq 0$ and $v_z\neq 0$, and this in turn implies that $\mathcal{M}_{(\rho_w,0,\theta_w)} = \mathcal{M}_{(1,0,1)}$, i.e. $\rho_w = \theta_w = 1$. This concludes the proof of Theorem 5.1. □

Proof of Lemma 5.2. By using the first relation in (5.2)

$$\begin{aligned} &\int_{v_z>0}\left[\Phi(v)\ln\left(\frac{\Phi(v)}{\mathcal{M}_{(1,0,1)}(v)}\right) - \Phi(v) + \mathcal{M}_{(1,0,1)}(v)\right]v_z\,dv \\ &\quad -\int_{v_z>0}\left[\mathcal{M}_{(\rho,0,\theta)}\ln\left(\frac{\mathcal{M}_{(\rho,0,\theta)}}{\mathcal{M}_{(1,0,1)}}\right) - \mathcal{M}_{(\rho,0,\theta)} + \mathcal{M}_{(1,0,1)}\right]v_z\,dv \\ &\quad = \int_{v_z>0}\left[\Phi(v)\ln\left(\frac{\Phi(v)}{\mathcal{M}_{(1,0,1)}(v)}\right) - \mathcal{M}_{(\rho,0,\theta)}\ln\left(\frac{\mathcal{M}_{(\rho,0,\theta)}}{\mathcal{M}_{(1,0,1)}}\right)\right]v_z\,dv. \end{aligned}$$

Using next the first and the fourth relations in (5.2), together with the formula $\ln\mathcal{M}_{(1,0,1)} = -\frac{3}{2}\ln(2\pi) - \frac{1}{2}|v|^2$, one further reduces the right hand side of the equality above to

$$\begin{aligned} &\int_{v_z>0}\left[\Phi(v)\ln\left(\frac{\Phi(v)}{\mathcal{M}_{(1,0,1)}(v)}\right) - \mathcal{M}_{(\rho,0,\theta)}\ln\left(\frac{\mathcal{M}_{(\rho,0,\theta)}}{\mathcal{M}_{(1,0,1)}}\right)\right]v_z\,dv \\ &\quad = \int_{v_z>0}\left(\Phi(v)\ln\Phi(v) - \mathcal{M}_{(\rho,0,\theta)}\ln\mathcal{M}_{(\rho,0,\theta)}\right)v_z\,dv. \end{aligned}$$

Further, since $\ln \mathcal{M}_{(\rho,0,\theta)}$ is also a linear combination of 1 and $|v|^2$, the same argument as in the case $\rho = \theta = 1$ shows that

$$\int_{v_z>0} \left(\Phi(v) - \mathcal{M}_{(\rho,0,\theta)}\right) \ln \mathcal{M}_{(\rho,0,\theta)} v_z dv = 0.$$

Hence, using also the first relation in (5.2), we finally arrive at

$$\begin{aligned}\int_{v_z>0} & \left(\Phi(v) \ln \Phi(v) - \mathcal{M}_{(\rho,0,\theta)} \ln \mathcal{M}_{(\rho,0,\theta)}\right) v_z dv \\ &= \int_{v_z>0} \left[\Phi \ln\left(\frac{\Phi}{\mathcal{M}_{(\rho,0,\theta)}}\right) - \Phi + \mathcal{M}_{(\rho,0,\theta)}\right] v_z dv.\end{aligned}$$

This establishes the equality in the conclusion of Lemma 5.2.

As for the positivity, it follows from the following elementary observation: for each a and $b \in (0, +\infty)$, one has

$$a \ln(a/b) - a + b \geq 0,$$

with equality if and only if $a = b$. Observe that the equality case here guarantees that the relation (5.2) define indeed a unique Maxwellian state.

Remark 1. As the proof shows, the same result holds if the Boltzmann equation is replaced by the BKW model, as in most of the numerical works on this question—see Sec. 2.

Remark 2. Theorem 5.1 can be extended to a general simple boundary, where there is no mass flux across the boundary (Sone[(23)]). The boundary condition is given as

$$\begin{aligned} F\big|_{z=0,\ v_z>0} &= \int_{v_{z*}<0} K(v, v_*) F(0, v_*) dv_*, \\ \lim_{z\to+\infty} F(z, v) &= \mathcal{M}_{(\rho_\infty, u_\infty, \theta_\infty)}, \end{aligned} \tag{5.3}$$

where the condition on the wall $z = 0$ is expressed with a scattering kernel $K(v, v_*)$. The scattering kernel $K(v, v_*)$ satisfies the conditions

(i) $K(v, v_*) \geq 0 \quad (v_z > 0, \quad v_{z*} < 0)$,
(ii) $\int_{v_z>0} \frac{v_z}{v_{z*}} K(v, v_*) \quad dv = -1 \quad (v_{z*} < 0)$,
(iii) $\mathcal{M}_{(\rho,0,\theta_w)} = \int_{v_{z*}<0} K(v, v_*) \mathcal{M}_{(\rho,0,\theta_w)} dv_* \quad (v_z > 0)$

where ρ is arbitrary and the other Maxwellians do not satisfy the condition (iii). Then, the solution F of the Boltzmann equation (1.2) subject to the boundary condition (5.3) exists only when $u_\infty = 0$ and $\theta_\infty = \theta_w$, and it is uniquely given by $F(z, v) = \mathcal{M}_{(\rho_\infty,0,\theta_w)}$.

This remark is used to show the uniqueness of the nonslip boundary condition for the fluid-dynamic-type equations in the continuum limit (see ref. 23).

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